Lecture 4

21 July 2022 17:45

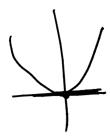
↳ Integer Linear Programs (ILP)

↳ Continuous variables ⟶ Stationary Points/ Critical points

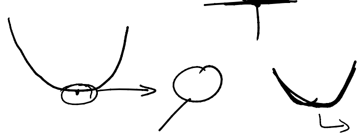
↳ Minimum/Maximum/ Point of inflexion

- $f(x) = x^2$

  ↳ Minimize $\quad x = 0 \quad f(0) = 0$



$f'(0) = 0 \rightarrow$ stationary point



$f''(x) = 2 > 0$

$f'(x^*) = 0,\ f''(x^*) > 0$

$\Downarrow$

x^* is a local minimizer.

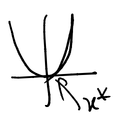
- $f(x) = -x^2$

  ↳ Maximize $\quad x^* = 0$

$\boxed{f'(0) = 0 \quad f''(0) < 0}$

$\Downarrow$ x^* is a local maximizer

- $f(x) = x^4$



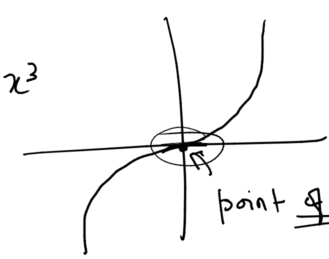
$x^* = 0$

$f'(0) = 0$, $f''(0) = 0$ → 0 is a stationary point

$\boxed{f'(0) = 0,\ f''(0) = 0}$

You cannot conclude minima/ maxima/ point of inflexion

- $f(x) = x^3$



point of inflexion

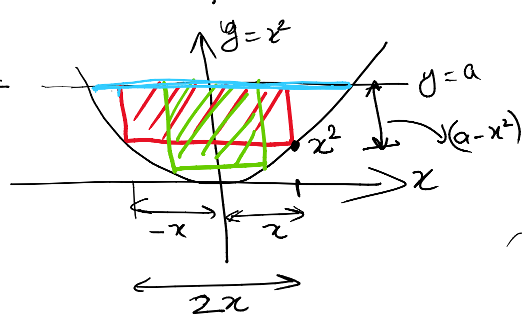
$f'(x) = 3x^2$

$f''(x) = 6x$

$f'(0) = 0$

$f''(0) = 0$

Ex:



Find the ~~rectangle~~ rectangle with max area.

$A = (2x)(a - x^2)$

$= 2(ax - x^3)$

$\frac{dA}{dx} = 0$ Stationary point

$$\frac{dA}{dx} = 0 \longrightarrow \text{Stationary point}$$

$$\underbrace{2(a - 3x^2)} = 0$$

$$\boxed{x^* = \sqrt{\frac{a}{3}}} \longrightarrow x^* \text{ is a critical point}$$

$$\left.\frac{dA}{dx}\right|_{x=x^*} = 0 \quad \text{at } x^* = \sqrt{\frac{a}{3}}$$

$$\frac{d^2A}{dx^2} = -12x \qquad \boxed{\left.\frac{d^2A}{dx^2}\right|_{x=x^*} < 0}$$

$\Rightarrow$ Area gets maximized at $x^* = \sqrt{\frac{a}{3}}$

$$A^* = 2\sqrt{\frac{a}{3}}\left(a - \frac{a}{3}\right) = \frac{4a\sqrt{a}}{3\sqrt{3}}$$

Ex: To produce x units of some product, a company spends,

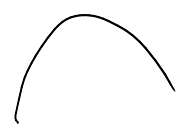
$$C(x) = ax^2 + bx \ (\$)$$

The product is sold at a price p \$ per unit. Determine the sales volume at which profit reaches maximum.

$$P(x) = R(x) - C(x)$$

$$P(x) = \underbrace{p.x - ax^2 - bx}$$

maximize



$$\frac{dP}{dx} = 0 \Rightarrow -2ax + p - b = 0$$

or $\boxed{x^* = \frac{p-b}{2a}}$

Stationary pt.

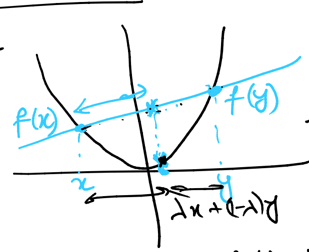
$$\frac{d^2P}{dx^2} < 0 \quad \text{at } x = x^*$$

$$-2a < 0 \Rightarrow x^* \rightarrow \frac{p-b}{2a}$$

Optimal number of units

Convexity (and concavity)

$f(x) = x^2$



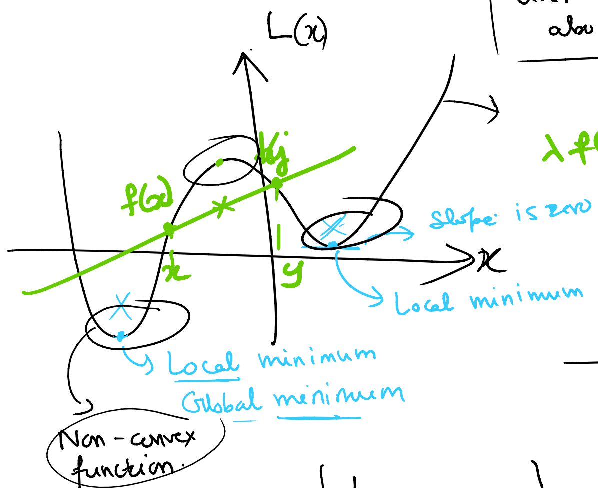
$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

Convex function

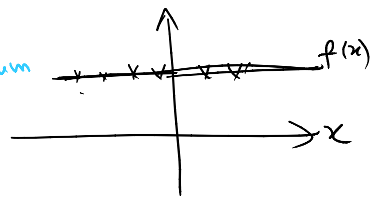
$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

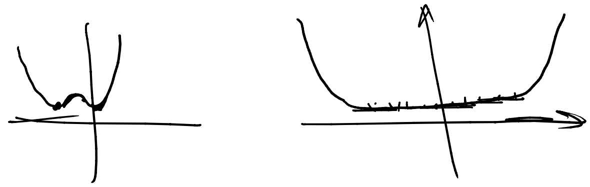
$$f(\lambda x+(1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

If a function is convex then every local minimum is also a global minimum



$$\lambda f(x) + (1-\lambda) f(y) < f(\lambda x + (1-\lambda)y)$$



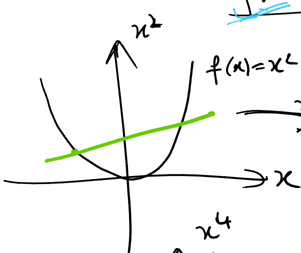


Convex function:

$$\boxed{f(\lambda x+(1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)}$$

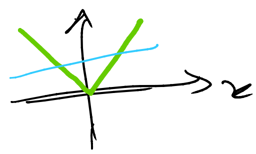
for every x, y

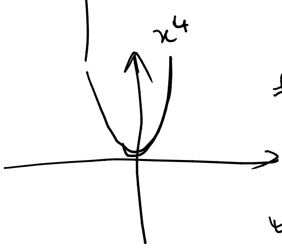
for every $\lambda \in (0,1)$



→ Convex function

$f''(x) = 2 > 0$

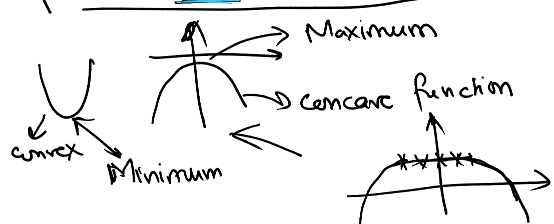




$f''(x) = 12x^2 \ge 0$

A function is convex if and only if $f''(x) \ge 0$

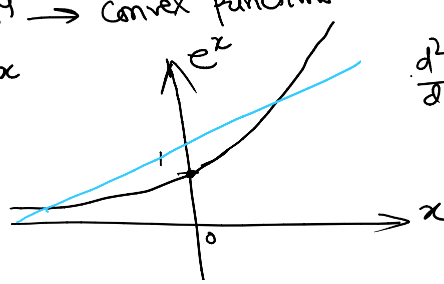
Concave functions:



* Examples of Convex functions:

$x^2, x^4 \rightarrow$ convex functions.

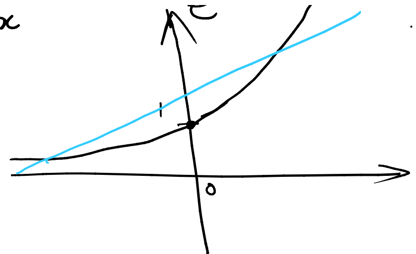
e^x



$\frac{d^2 e^x}{dx^2} = e^x > 0$

↳ Convex function

e^x



$\frac{d^2 e^x}{dx^2} = e^x > 0$

↳ Convex function

$\log x$

$\log x$ is <u>concave</u>

$f(x) = \log x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0$

* Operations that preserve convexity:

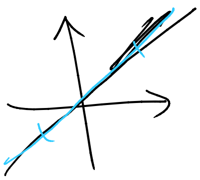
$f_1 \rightarrow$ convex

$f_2 \rightarrow$ convex

Non-negative weighted sum

$a f_1 + b f_2 := f$ $a, b > 0$

↓ Convex



$f_1(x) = x$

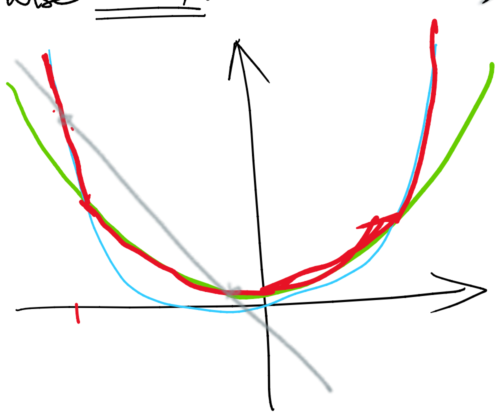
$f_2(x) = x^2$

$a f_1(x) + b f_2(x)$

$ax + bx^2$ is also convex for $a, b > 0$

$a = 1, \ b = -1$ $x - x^2$

* Pointwise Maximum: → Preserves convexity



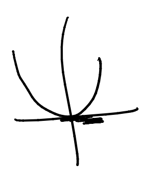
$f(x) = \max\{f_1(x), f_2(x)\}$

↳ Convex

* Function of several variables:

$f'(x^*) = 0 \rightarrow$ Stationary pt.

$f''(x^*) > 0$ — x^* is a minimizer



$f''(x^*) > 0 \longrightarrow x^*$ is a minimizer

$$f(x,y) = 3x^2y + xy + y^2x + 3y$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad \nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

→ Gradient of f

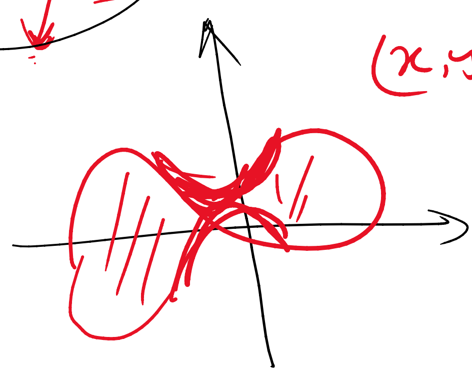
$$\nabla f = \begin{bmatrix} 6xy + y + y^2 \\ 3x^2 + x + 2y + 3 \end{bmatrix} = 0$$

$$f(x) = x^2 - y^2$$

$(x^*, y^*) \Rightarrow \nabla f(x^*, y^*) = 0$

→ Critical points or Stationary points

$(x,y) = (0,0)$



$$\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix} = 0 \Rightarrow (x^*, y^*) = (0,0)$$

$$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Second order condition: → We compute Hessians.

$f_{xx}, \ f_{xy}, \ f_{yx}, \ f_{yy}$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 6xy + y + y^2 \\ 3x^2 + x + 2y + 3 \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} 6y & 6x+1+2y \\ 6x+1+2y & 2 \end{bmatrix}$$ → Symmetric Matrix

... then x^* is a local min

$6x+1$

$\nabla f = 0$ and $\nabla^2 f$ is positive definite, then x^* is a local min

$\nabla^2 f$ is negative " x^* is a local max

$\nabla^2 f$ is neither +ve nor −ve definite, then we cannot conclude about x^*.

$f_{xx} > 0$ and Determinant is $> 0 \Rightarrow$ Matrix is positive definite

$f_{xx} < 0$ and Determinant is $> 0 \Rightarrow$ Matrix is negative "

and if Determinant $\leq 0 \Rightarrow$ we cannot conclude.

$A_{n \times n}$ $\quad x \in \mathbb{R}^n$ $\quad \underbrace{x^T}_{1\times n} A \underbrace{x}_{n \times 1}$ → $x^T A x > 0$ for every $x \neq 0$

$x^T A x < 0$ for every $x \neq 0$

→ Negative definite

$x^2 = x \cdot 1 \cdot x$

$-x^2 = x(-1) \cdot x$